University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PRACTICE FINAL EXAMINATION

Directions. Do all six problems (weights are indicated). This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (30 points)
A mass m is connected to a wall by a spring of constant k. Define $\omega_0 \equiv \sqrt{k/m}$. There is no damping force.
(**a**) (20 points)
In addition to the spring force, the mass is subjected to an external force $F_{\rm ext}$:

$$\begin{aligned} F_{\rm ext} &= 0, \quad t < 0 \\ &= F_0 \sin 2\omega_0 t, \quad 0 < t < 2\pi/\omega_0 \\ &= 0, \quad 2\pi/\omega_0 < t \ , \end{aligned}$$

where F_0 and ω_0 are constants. Find $x(t)$ for $t > 2\pi/\omega_0$.
(*Hint*: A solution to the differential equation

$$\Big(\frac{d^2}{dt^2} + \omega_0^2\Big) G(t) = \delta(t) \ ,$$

where $\delta(t)$ is a Dirac delta function, and G and $\dot G$ vanish for $t \le 0$, is

$$\begin{aligned} G(t) &= 0, \quad t \le 0 \\ &= \frac{\sin \omega_0 t}{\omega_0}, \quad t > 0 \ .) \end{aligned}$$

Solution:
The equation given for $G(t)$ defines it to be a Green function. Correspondingly, the solution for $x(t)$ is

$$x(t) = \int_{-\infty}^{t} \frac{F_{\rm ext}(t')}{m} G(t-t')\, dt'$$

$$x(t > \tfrac{2\pi}{\omega_0}) = \int_0^{\frac{2\pi}{\omega_0}} \frac{F_0}{m} \sin 2\omega_0 t' \frac{\sin \omega_0 (t-t')}{\omega_0}\, dt'$$

$$= \int_0^{\frac{2\pi}{\omega_0}} \frac{F_0}{m} \sin 2\omega_0 t' \, \tfrac{\sin\omega_0 t \cos\omega_0 t' - \sin\omega_0 t' \cos\omega_0 t}{\omega_0}\, dt'$$

The integral in the last line splits into two pieces which are proportional, respectively, to

$$\int_0^{\frac{2\pi}{\omega_0}} \sin 2\omega_0 t' \cos\omega_0 t'\, dt' \ \text{ and}$$
$$\int_0^{\frac{2\pi}{\omega_0}} \sin 2\omega_0 t' \sin\omega_0 t'\, dt' \ .$$

The first piece vanishes because $\sin 2\omega_0 t'$ is odd and $\cos\omega_t'$ is even with respect to the midpoint of the interval; the second piece vanishes because $\sin my$ and $\sin ny$ are orthogonal functions when $m \ne n$. Therefore

$$x\big(t > \tfrac{2\pi}{\omega_0}\big) = 0 \ .$$

(**b**) (10 points)
As an alternative to applying an external force $F_{\rm ext}$, this oscillator could be excited by causing the spring "constant" k to vary sinusoidally with time:

$$k(t) = k_0(1 + \epsilon_0 \cos\Omega t) \ ,$$

where ϵ_0 and Ω are constants. If such a variation were to occur for a long time, even if $\epsilon_0 \ll 1$, certain value(s) of Ω would cause the mass to oscillate with an amplitude that grows exponentially with time. Can you provide an example of such a value for Ω? (Here you are asked merely to recall a result from the reading and classroom discussion in which you have participated.)

Solution:
This is the Mathieu equation describing *e.g.* a child pumping a swing. Its solution exhibits a parametric resonance at

$$\Omega = 2\omega_0 \ .$$

2. (20 points)
A mass m is in uniform circular motion at constant radius R about a center of attractive force

$$\mathbf{F}(r) = -\frac{K\hat{r}}{r^2},$$

where K is a positive constant. The mass receives a slight nudge, causing the radius of its orbit to acquire a small perturbed component, so that

$$r(t) = R(1 + \epsilon \cos \omega' t),$$

where ϵ is a constant $\ll 1$. Find the angular frequency ω' of the perturbation. Justify your answer either by explicit calculation, or by simple arguments based on your knowledge of the orbit.
Solution:
A $1/r^2$ attractive central force yields an elliptical bound orbit, with the focus of the ellipse located at the center of force. Relative to its focus, an ellipse has one fixed point of maximum radius and one fixed point of minimum radius. Therefore the angular frequency with which the radius varies is the same as the angular frequency ω_0 of the basic orbit. From centrifugal force balance for the (unperturbed) circular orbit, this is

$$\begin{aligned} \frac{mv^2}{R} &= \frac{K}{R^2} \\ v &= \sqrt{\frac{K}{mR}} \\ \omega_0 &= \frac{v}{R} = \sqrt{\frac{K}{mR^3}} \ . \end{aligned}$$

The same solution can be obtained with more effort by evaluating the effective spring constant

$$k_{\text{eff}} \equiv \left. \frac{d^2 U_{\text{eff}}}{dr^2} \right|_{\text{min}} ,$$

where the effective potential U_{eff} is the sum of $l^2/2mr^2$ and the potential that yields $\mathbf{F}(r)$; then $\omega = \sqrt{k_{\text{eff}}/m}$. As a second alternative, the method of perturbations may be applied.

3. (35 points)
A square thin metal plate has area b^2 and mass m. A set of body axes is set up with the origin at the CM of the plate. $\hat{z}$ is normal to the plate, while $\hat{x}$ and $\hat{y}$ intersect the plate's *corners*. At $t = 0$ the angular velocity of the plate is

$$\vec{\omega}(0) = \frac{\omega_0}{\sqrt{2}}(\hat{x} + \hat{z}) \ .$$

(**a**) (15 points)
At $t = 0$, compute the angular momentum $\vec{L}(0)$ (measured in the body system).
Solution:
First we need to compute the inertia tensor $\mathcal{I}$ of the plate. For the moment, imagine that the $\hat{x}$ and $\hat{y}$ axes pass through the *sides* rather than the *corners* of the plate. Then $\mathcal{I}_{xx}$ and $\mathcal{I}_{yy}$ would be easy to compute – either would be equal to $\frac{1}{12}mb^2$, the moment of inertia of a stick. By symmetry, $\hat{x}$ and $\hat{y}$ would be principal axes, as is $\hat{z}$. Since the (thin) plate is a plane,

$$\mathcal{I}_{zz} = \mathcal{I}_{xx} + \mathcal{I}_{yy} = \tfrac{1}{6}mb^2 \ .$$

How is the actual situation different, given that the $\hat{x}$ and $\hat{y}$ axes actually pass through the plate's corners? Not at all. Still, by symmetry, $\hat{x}$ and $\hat{y}$ are principal axes. The above equation still requires $\mathcal{I}_{xx}$ and $\mathcal{I}_{yy}$ to have the same value. Therefore

$$\mathcal{I} = \tfrac{1}{12}mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \ .$$

Finally

$$\begin{aligned} \vec{L} &= \mathcal{I}\vec{\omega} \\ &= \tfrac{1}{12}mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \frac{\omega_0}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{mb^2\omega_0}{12\sqrt{2}}(\hat{x} + 2\hat{z}) \ . \end{aligned}$$

(**b**) (15 points)
The motion of the plate is allowed to evolve freely, without the influence of any external forces or torques. At what time will $\vec{L}$, as measured in the body system, be directed within the $\hat{y}$-$\hat{z}$ plane rather than the $\hat{x}$-$\hat{z}$ plane?
Solution:
For ease of notation, define $\hat{x}$, $\hat{y}$, and $\hat{z}$ to be the "1", "2", and "3", directions, respectively. Euler's equations

$$\begin{aligned} \mathcal{I}_{33}\dot{\omega}_3 - (\mathcal{I}_{11} - \mathcal{I}_{22})\omega_1\omega_2 &= N_3 \\ \mathcal{I}_{11}\dot{\omega}_1 - (\mathcal{I}_{22} - \mathcal{I}_{33})\omega_2\omega_3 &= N_1 \\ \mathcal{I}_{22}\dot{\omega}_2 - (\mathcal{I}_{33} - \mathcal{I}_{11})\omega_3\omega_1 &= N_2 \end{aligned}$$

become

$$\begin{aligned} \dot{\omega}_3 &= 0 \\ \dot{\omega}_1 &= -\omega_2\omega_3 \\ \dot{\omega}_2 &= +\omega_1\omega_3 \ . \end{aligned}$$

Therefore ω_3 is a constant, and $\vec{\omega}_\perp$, the component of $\vec{\omega}$ that is $\perp$ to $\hat{z}$, rotates around $\hat{z}$ with angular velocity ω_3. So in $\frac{1}{4}$ of a period, or

$$\Delta t = \frac{\pi}{2\omega_3} = \frac{\pi}{\sqrt{2}\omega_0} \ ,$$

the angular velocity will rotate into the $\hat{x}$-$\hat{z}$ plane. By the results of (**a.**), so will the angular momentum.
(**c**) (5 points)
In the absence of external torques, angular momentum is conserved. Does this fact conflict with your answer to part (**b.**)? Explain.
Solution:
No contradiction is implied. Angular momentum is conserved only in an *inertial* system, where Newton's laws hold. In part (**b.**) we calculated the evolution of angular momentum in the *body* system, which is rotating and therefore not inertial.

4. (35 points)
(**a**) (7 points)
A compact disk ("CD") of mass m and radius b is suspended from its center by a strictly vertical wire of torsional constant γ. (Ignore the wire's mass and the CD's hole.) The disk remains horizontal and is free only to twist (with azimuthal angle φ) in the horizontal plane, such that the potential energy stored in the twisted wire is

$$U(\psi) = \tfrac{1}{2}\gamma\varphi^2.$$

Find the frequency ω_a of small oscillations of φ.
Solution:
The disk's moment of inertia is

$$I = \frac{m}{\pi b^2}\int_0^b r^2 2\pi r\, dr = \tfrac{1}{2}mb^2$$

(it can be recalled rather than calculated). The Lagrangian is

$$\mathcal{L} = \tfrac{1}{4}mb^2\dot{\varphi}^2 - \tfrac{1}{2}\gamma\varphi^2 \ .$$

The Euler-Lagrange equation yields

$$\tfrac{1}{2}mb^2\ddot{\varphi} + \gamma\varphi = 0 \ .$$

This is the usual equation for simple harmonic motion with angular frequency

$$\omega_a = \sqrt{\frac{2\gamma}{mb^2}} \ .$$

(**b**) (7 points)
The system is now made more complicated by the addition of a second identical CD, suspended from the first CD by a second identical torsion wire. Take ψ to be the azimuthal angle by which CD #2 is twisted – its full twist, not its twist relative to CD #1. Thus the net amount by which wire #2 is twisted is $(\psi - \varphi)$. Using φ and ψ as generalized coordinates, find the 2×2 symmetric matrix $\mathcal{M}$ such that the kinetic energy T is given by

$$T = \tfrac{1}{2}\begin{pmatrix}\dot{\varphi} & \dot{\psi}\end{pmatrix}\mathcal{M}\begin{pmatrix}\dot{\varphi} \\ \dot{\psi}\end{pmatrix} \ .$$

Solution:
In analogy to the single disk system, the kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{4}mb^2\dot{\varphi}^2 + \tfrac{1}{4}mb^2\dot{\psi}^2 \\ &= \tfrac{1}{4}mb^2\begin{pmatrix}\dot{\varphi} & \dot{\psi}\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\dot{\varphi} \\ \dot{\psi}\end{pmatrix} \ . \end{aligned}$$

Thus

$$\mathcal{M} = \tfrac{1}{2}mb^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

(**c**) (7 points)
For the same system, find the 2×2 symmetric matrix $\mathcal{K}$ such that the potential energy U is given by

$$U = \tfrac{1}{2} \begin{pmatrix} \varphi & \psi \end{pmatrix} \mathcal{K} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} .$$

Solution:
In analogy to the single disk system, the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}\gamma\varphi^2 + \tfrac{1}{2}\gamma(\psi - \varphi)^2 \\ &= \tfrac{1}{2}\gamma(2\varphi^2 + \psi^2 - \varphi\psi - \psi\varphi) \\ &= \tfrac{1}{2}\gamma \begin{pmatrix} \varphi & \psi \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} . \end{aligned}$$

Thus

$$\mathcal{K} = \gamma \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} .$$

(**d**) (7 points)
Obtain the natural angular frequencies of oscillation of this system.
Solution:
The secular equation for the natural frequencies ω is

$$\begin{aligned} 0 &= \det(\mathcal{K} - \omega^2\mathcal{M}) \\ &= \left| \frac{2\gamma}{mb^2} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} - \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \\ &\equiv \left| \omega_a^2 \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} - \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| . \end{aligned}$$

Defining $\eta \equiv \omega^2/\omega_a^2$,

$$\begin{aligned} 0 &= \det \begin{pmatrix} 2-\eta & -1 \\ -1 & 1-\eta \end{pmatrix} \\ &= 2 - 3\eta + \eta^2 - 1 \\ \eta &= \tfrac{3}{2} \pm \sqrt{\tfrac{9}{4} - 1} \\ &= \tfrac{3}{2} \pm \sqrt{\tfrac{5}{4}} \\ \omega^2 &= \tfrac{1}{2}\omega_a^2(3 \pm \sqrt{5}) \\ &= \frac{\gamma}{mb^2}(3 \pm \sqrt{5}) . \end{aligned}$$

(**e**) (7 points)
Describe the motion of the two CDs when the system has only one normal mode excited (you may choose any mode you wish). Your description should specify the amplitude of ψ relative to that of φ.
Solution:
The equation that yields the (unnormalized) eigenvectors is

$$0 = \begin{pmatrix} 2-\eta & -1 \\ -1 & 1-\eta \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} .$$

The top line of this pair of equations is

$$0 = (2-\eta)\varphi - \psi .$$

Substituting *e.g.* $\eta = \frac{1}{2}(3 - \sqrt{5})$ for the slower mode,

$$\psi = \frac{1+\sqrt{5}}{2}\varphi ,$$

so the lower CD twists in phase with the upper CD at $\approx 162\%$ of the upper CD's amplitude.

5. (40 points)
A *spherical top* of mass m under the influence of gravity with one point fixed is described by the usual Euler angles ϕ (= azimuth of the top's axis about the vertical), θ (= polar angle of the top's axis with respect to vertical), and ψ (= azimuth of the top about its axis). Gravity pulls down on the top's CM, which is a distance h from the (frictionless) pivot. The top's kinetic energy is given by

$$T = \tfrac{1}{2}I(\dot\phi^2 + \dot\theta^2 + \dot\psi^2 + 2\dot\phi\dot\psi\cos\theta) ,$$

where I is the top's moment of inertia about its symmetry axis, and also its moment of inertia about any axis which is perpendicular to the symmetry axis and which passes through the pivot. (The fact that there is only a single moment of inertia I is the reason that this top is called "spherical". Since I is measured about the pivot, not about the CM, the top itself is not spherically symmetric.)

(**a**) (20 points)
A generalized force of constraint Q_ϕ (actually a torque about the vertical axis) is applied to the top so that ϕ is constrained to be constant. For the initial conditions $\theta(0) \equiv \theta_0 \ll 1$, $\dot{\theta}(0) = 0$, and $\dot{\psi}(0) = \omega_0$, solve for $\theta(t)$ in the regime $\theta \ll 1$.
Solution:
The Lagrangian for the spherical top is

$$\mathcal{L} = \tfrac{1}{2} I(\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta) - mgh\cos\theta \ .$$

The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \frac{\partial L}{\partial \phi} + Q_\phi \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= \frac{\partial L}{\partial \theta} \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} &= \frac{\partial L}{\partial \psi} \ . \end{aligned}$$

Applying these equations to the above Lagrangian,

$$\begin{aligned} I(\ddot{\phi} + \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta) &= Q_\phi \\ I\ddot{\theta} &= (mgh - I\dot{\phi}\dot{\psi})\sin\theta \\ I(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta) &= 0 \ . \end{aligned}$$

Enforcing the constraint $\phi =$ constant, so that $\dot{\phi} = \ddot{\phi} = 0$, these equations simplify to

$$\begin{aligned} I(\ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta) &= Q_\phi \\ I\ddot{\theta} &= mgh\sin\theta \\ I\ddot{\psi} &= 0 \ . \end{aligned}$$

Only the second Euler-Lagrange equation is needed to solve this part of the problem. Approximating $\sin\theta \approx \theta$,

$$\begin{aligned} I\ddot{\theta} &= mgh\theta \\ \theta &= \theta_0 \cosh\sqrt{\tfrac{mgh}{I}}\,t \ . \end{aligned}$$

This is the familiar result for a falling stick; it is the same result as would be obtained if the top were not spinning at all.
(**b**) (20 points)
For the conditions of (**a.**), find the generalized force of constraint $Q_\phi(t)$ which must be exerted upon the top to keep $\phi =$ constant.
Solution:
From the third Euler-Lagrange equation, $\ddot{\psi}$ vanishes, so $\dot{\psi} = \omega_0$ for all time. The first Euler-Lagrange equation simplifies to

$$\begin{aligned} -I\omega_0\dot{\theta}\sin\theta &= Q_\phi \\ -I\omega_0\dot{\theta}\theta &\approx Q_\phi \ , \end{aligned}$$

where again we have made the small-angle approximation $\sin\theta \approx \theta$. Substituting $\theta(t)$ from part (**a.**),

$$\begin{aligned} &Q_\phi(t) = \\ &= -I\omega_0\theta_0^2\sqrt{\tfrac{mgh}{I}}\ \sinh\left(\sqrt{\tfrac{mgh}{I}}t\right)\ \cosh\left(\sqrt{\tfrac{mgh}{I}}t\right) \\ &= -\tfrac{1}{2}I\omega_0\theta_0^2\sqrt{\tfrac{mgh}{I}}\ \sinh\left(2\sqrt{\tfrac{mgh}{I}}t\right) \ . \end{aligned}$$

6. (40 points)
Consider a long narrow rectangular membrane which, in equilibrium, lies in the x-y plane; $\hat{x}$ is its long direction and $\hat{y}$ is its short direction. The membrane's displacement normal to the x-y plane is denoted by $z(x, y, t)$. The membrane is clamped at its long edges $y = 0$ and $y = b$, so that

$$z(x, 0, t) = z(x, b, t) = 0 \ .$$

We wish to investigate the propagation of traveling sinusoidal waves $z(x, y, t)$ in the long direction $\hat{x}$.
(**a**) (6 points)
The Lagrangian density $\mathcal{L}'$ (per unit area of membrane) is given by

$$\begin{aligned} &\mathcal{L}'(z, \tfrac{\partial z}{\partial x}, \tfrac{\partial z}{\partial y}, \tfrac{\partial z}{\partial t}, x, y, t) = \\ &= \tfrac{1}{2}\sigma\left(\tfrac{\partial z}{\partial t}\right)^2 - \tfrac{1}{2}\beta\Big(\left(\tfrac{\partial z}{\partial x}\right)^2 + \left(\tfrac{\partial z}{\partial y}\right)^2\Big) \ , \end{aligned}$$

where σ is the membrane's mass per unit area, and β is a positive constant that is inversely proportional to its elasticity. Apply the Euler-Lagange equations to this Lagrangian density to obtain a partial differential equation for $z(x, y, t)$.

Solution:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\frac{\partial z}{\partial t})}+\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial(\frac{\partial z}{\partial x})}+\frac{d}{dy}\frac{\partial\mathcal{L}}{\partial(\frac{\partial z}{\partial y})}=\frac{\partial\mathcal{L}}{\partial z}$$
$$\sigma\frac{\partial^2 z}{\partial t^2}-\beta(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2})=0\ .$$

(**b**) (6 points)
Search for a trial solution in the form

$$z(x,y,t)=Y(y)\cos\left(k_x x-\omega t\right)\ ,$$

where $Y(y)$ is a function only of y, and k_x and ω are constants which for the moment are unspecified. Plug this solution into the equation you obtained for (**a.**). Dividing through by $\cos\left(k_x x-\omega t\right)$, obtain an ordinary differential equation for $Y(y)$.
Solution:

$$(-\sigma\omega^2 Y+\beta k_x^2 Y-\beta Y'')\cos\left(k_x x-\omega t\right)=0$$
$$\beta Y''+Y(\sigma\omega^2-\beta k_x^2)=0\ .$$

(**c**) (6 points)
Applying the boundary conditions $z(x,0,t)=z(x,b,t)=0$, identify and choose a (non null) solution for $Y(y)$ which has the most gradual dependence on y that is possible given these conditions.
Solution:
The general solution for Y will be a sum of $\sin\sqrt{\sigma\omega^2-\beta k_x^2}y$ and $\cos\sqrt{\sigma\omega^2-\beta k_x^2}y$ if $\sigma\omega^2>\beta k_x^2$, or a sum of $\sinh\sqrt{\beta k_x^2-\sigma\omega^2}y$ and $\cosh\sqrt{\beta k_x^2-\sigma\omega^2}y$ if $\sigma\omega^2<\beta k_x^2$. However, no sum of sinh and cosh can vanish at both $y=0$ and $y=b$. Therefore $\sigma\omega^2>\beta k_x^2$ and we have a sum of sin and cos. In order to vanish at $y=0$ and $y=b$, the cos part must vanish, and

$$Y(y)\propto\sin\frac{n\pi y}{b}\ ,$$

where n is a positive integer. The most gradual dependence on y occurs when $n=1$. Therefore

$$Y(y)\propto\sin\frac{\pi y}{b}\ .$$

(**d**) (6 points)
Returning to the equation you obtained for (**a.**), plug in your answer to (**c.**) to obtain an equation relating k_x and ω.
Solution:
Adopting the above solution for Y,

$$\frac{\beta\pi^2}{b^2}=\sigma\omega^2-\beta k_x^2$$
$$\omega^2=\frac{\beta}{\sigma}\left(k_x^2+\frac{\pi^2}{b^2}\right)\ .$$

(**e**) (6 points)
What is the minimum frequency $\omega_{\rm min}$ of sinusoidal waves that can propagate in the $\hat{x}$ direction without attenuation?
Solution:
For a sinusoidal wave to propagate in the $\hat{x}$ direction without attenuation, k_x must be real, so that $k_x^2>0$. Then from the previous equation

$$\omega_{\rm min}=\frac{\pi}{b}\sqrt{\frac{\beta}{\sigma}}\ .$$

(**f**) (5 points)
If $\omega=\sqrt{2}\omega_{\rm min}$, calculate the *phase* velocity with which sinusoidal waves propagate in the $\hat{x}$ direction.
Solution:
If $\omega=\sqrt{2}\omega_{\rm min}$, from (**d.**) $k_x^2=\pi^2/b^2$. Then

$$v_{\rm phase}=\frac{\omega}{k_x}=\frac{\sqrt{2}(\pi/b)\sqrt{\beta/\sigma}}{\pi/b}=\sqrt{\frac{2\beta}{\sigma}}\ .$$

(**g**) (5 points)
What is the *group* velocity of the waves described in (**e.**)?
Solution:

$$v_{\rm group}=\frac{d\omega}{dk_x}\ .$$

From (**d.**),

$$\omega\, d\omega=\frac{\beta}{\sigma}k_x\, dk_x\ .$$

Therefore

$$\begin{aligned} v_{\text{group}} &= \frac{\beta}{\sigma}\frac{k_x}{\omega} \\ &= \frac{\beta}{\sigma v_{\text{phase}}} \\ &= \sqrt{\frac{\beta}{2\sigma}}\,. \end{aligned}$$

Note that the geometric mean of v_{phase} and v_{group} remains equal to $\sqrt{\frac{\beta}{\sigma}}$, the "pure" phase velocity of waves traveling on the membrane when no boundary restrictions are applied. (This problem is simpler than, but similar to, the problem of EM wave propagation in a hollow rectangular waveguide.)